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NON-LOCAL CRITERIA FOR THE EXISTENCE AND STABILITY OF PERIODIC OSCILLATIONS IN AUTONOMOUS HAMILTONIAN SYSTEMS*

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The conditions under which single-parameter families of periodic solutions (the existence in a sufficiently small neighbourhood of the origin of coordinates follows from the Lyapunov theorem (see /1/)) can be continued in a parameter to the boundary of the given domain, in particular to a certain isoenergetic surface, are found. These conditions, which can be verified by the use of the Hessian of a Hamilton function, also ensure the orbital stability of solutions to a first approximation. Bilateral estimates of the oscillation periods are obtained, and it is established that any solution with a period which satisfies such an estimate belongs to the corresponding family. As an example, the non-linear oscillations of a string with lumped masses are examined.

The well-known non-local results relevant to the periodic oscillations of autonomous Hamiltonian systems are, as a rule, theorems on the existence of periodic solutions (see reviews /2—4/). One group of papers establishes the existence of periodic solutions with a specified value of the Hamiltonian, and other papers, establish solutions with a specified period; in the first case assumptions and made regarding the form of the corresponding constant energy surface; and in the second assumptions are made regarding the behaviour of the Hamiltonian in the vicinity of the equilibrium configuration and at infinity. The majority of the results were obtained by variational methods, the desired periodic solutions being identified with the stationary points of certain functionals. The discussion in the present paper is based on other concepts.

1. Consider the system.

$$x_i^{\cdot}=\frac{\partial H}{\partial x_{i+n}},\quad x_{i+n}^{\cdot}=-\frac{\partial H}{\partial x_i},\quad i=1,\ldots,n \tag{1.1}$$

where $x_1,\ldots,x_n$ and $x_{n+1},\ldots,x_{2n}$ are the generalized coordinates and momenta, and $H(x_1,\ldots,x_{2n})$ is the Hamiltonian function, doubly differentiable with respect to x_i.

Let $\mathbf{x}^\circ(t)=(x_1^\circ(t),\ldots,x_{2n}^\circ(t))'$ be a periodic solution of system (1.1) with period T_0 (here the prime denotes transposition). The corresponding variational equation is

$$\begin{gathered}J\mathbf{y}^{\cdot}=A_0(t)\,\mathbf{y}\\ A_0(t)=\|a_{ik}^\circ(t)\|_1^{2n},\quad a_{ik}^\circ=\left.\frac{\partial^2 H}{\partial x_i\partial x_k}\right|_{\mathbf{x}=\mathbf{x}^\circ(t)}\\ J=\begin{Vmatrix}0 & -I_n\\ I_n & 0\end{Vmatrix},\quad \mathbf{y}=(y_1,\ldots,y_{2n})'\end{gathered} \tag{1.2}$$

where I_n denotes the unit matrix of order n.

We will recall some well-known facts. System (1.1) admits of the integral

$$H(x_1(t),\ldots,x_{2n}(t))=\mathrm{const}, \tag{1.3}$$

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Let $Y(t)$ be a matrizant, and ρ_i $(i = 1, \ldots, 2n)$ the multiplicators of Eq.(1.2), i.e. the eigenvalues of the monodromy matrix $Y(T_0)$. Because system (1.1) is autonomous, Eq.(1.2) has a periodic solution $\mathbf{y}^\circ(t) = \mathbf{x}^{\circ\cdot}(t)$, the multiplicator $\rho = 1$ corresponds to this solution. Since (1.2) is canonical, the multiplicity of this multiplicator is $k \geqslant 2$ (incidentally, this deduction also holds for an autonomous system of general form, which has a first integral). The case of $k = 2$ is 'typical' while that of $k > 2$ is realized for some values of H.

The vector-function $\mathbf{z}^\circ(t) = (\partial H/\partial x_1, \ldots, \partial H/\partial x_{2n})'|_{\mathbf{x}=\mathbf{x}_0(t)}$ is a periodic solution (see /1/) of the adjoint equation

$$\mathbf{z}^\cdot = -(J^{-1}A_0(t))'\mathbf{z} \tag{1.4}$$

Differentiation of (1.3) with $\mathbf{x} = \mathbf{x}^\circ(t)$ yields the identity

$$(\mathbf{z}^\circ(t),\ \mathbf{y}^\circ(t)) \equiv 0 \tag{1.5}$$

As a rule, the closed trajectories of the autonomous Hamiltonian systems are not isolated: they form single-parameter families. The following auxiliary theorem gives sufficient conditions for the solution $\mathbf{x}^\circ(t)$ to be a member of such a family.

Theorem 1. If one Jordan block matrix or its multiplicity $k = 2$ corresponds to the multiplicator $\rho = 1$, then for sufficiently small $|s|$ system (1.1) has a unique single-parameter family of solutions $\mathbf{x}(t, s)$ such that $\mathbf{x}(t, 0) = \mathbf{x}^\circ(t)$.

Proof. Let $x_i(t, \alpha_1, \ldots, \alpha_{2n})$ be a solution of system (1.1) which satisfies the initial conditions $x_i(0, \alpha_1, \ldots, \alpha_{2n}) = \alpha_i$ $(i = 1, \ldots, 2n)$. If, for certain $\alpha_1, \ldots, \alpha_{2n}$ and T the equations

$$x_i(T, \alpha_1, \ldots, \alpha_{2n}) = \alpha_i,\ i = 1, \ldots, 2n \tag{1.6}$$

hold, then the corresponding solution $x_i(t, \alpha_1, \ldots, \alpha_{2n})$ is periodic with period T.

Since system (1.1) is autonomous, one of the quantities can be regarded as known. To be specific, we take $\alpha_{2n} = C$, choosing C such that

$$y_{2n}^\circ(0) \neq 0,\quad z_{2n}^\circ(0) \neq 0 \tag{1.7}$$

Since $\mathbf{x}^\circ(t) = \mathbf{x}^\circ(t + T_0)$, when $T = T_0$, $\alpha_i = x_i^\circ(0)$, $\alpha_{2n} = C$ Eqs.(1.6) hold. If for some α_i and T sufficiently close to $x_i^\circ(0)$ and T_0, then $2n - 1$ equalities (1.6) are satisfied, and by virtue of (1.3) and the second condition (1.7), we have $x_{2n}(T, \alpha_1, \ldots, \alpha_{2n-1}, C) = C$, that is the last equation in (1.6) is also satisfied.

As we know, $2n - 1$ Eqs.(1.6) determine, for sufficiently small $|s|$, a unique single-parameter family $T(s), \alpha_1(s), \ldots, \alpha_{2n-1}(s)$, which for $s = 0$ becomes T_0, $x_1^\circ(0), \ldots, x_{2n-1}(0)$ if the rank of the corresponding Jacobian matrix,

$$B = \left\| \begin{matrix} \dfrac{\partial x_1}{\partial t} & \dfrac{\partial x_1}{\partial \alpha_1} - 1 & \ldots & \dfrac{\partial x_1}{\partial \alpha_{2n-1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial x_{2n-1}}{\partial t} & \dfrac{\partial x_{2n-1}}{\partial \alpha_1} & \ldots & \dfrac{\partial x_{2n-1}}{\partial \alpha_{2n-1}} - 1 \end{matrix} \right\|_{\alpha_i = x_i^\circ(0),\ T = T_0}$$

equals $2n - 1$.

First we assume that the Jordanian block matrix corresponds to the multiplicator $\rho = 1$ i.e. Eq.(1.2) has a unique T_0-periodic solution $\mathbf{y}^\circ(t)$ (accurate to within a multiplier). We shall show that in this case the matrix B_1 obtained from B by crossing out the first column, is not singular and, therefore, the rank $B = 2n - 1$.

Let us assume that $\det B_1 = 0$; then the equation $B_1\mathbf{y} = 0$ has the non-trivial solution $(y_1, \ldots, y_{2n-1})'$. As we know,

$$Y(T_0) = \left\| \frac{\partial x_i(T, \alpha_1, \ldots, \alpha_{2n})}{\partial \alpha_k} \right\|_1^{2n} (\alpha_i = x_i^\circ(0), T = T_0)$$

and therefore matrix B_1 can be expressed in the form $B_1 = Y_{2n-1,2n-1}(T_0) - I_{2n-1}$, where $Y_{2n-1,2n-1}(T_0)$ is the matrix obtained from $Y(T_0)$ by crossing out the last column and the last row. Consequently, for $\mathbf{y}^1(y_1, \ldots, y_{2n-1}, 0)'$ the first $2n - 1$ components of the vector $Y(T_0)\mathbf{y}^1$ are $y_1, \ldots, y_{2n-1}$. Let $\mathbf{y}^1(t)$ be a solution of Eq.(1.2), which satisfies the condition $\mathbf{y}^1(0) = \mathbf{y}^1$, that is $\mathbf{y}^1(t) = Y(t)\mathbf{y}^1$. Obviously, the solution $\mathbf{z}^\circ(t)$ of the conjugate system (1.4) and any solution $\mathbf{y}(t)$ of system (1.2) also satisfy the relation

$$(\mathbf{z}^\circ(t),\ \mathbf{y}(t)) \equiv \text{const} \tag{1.8}$$

Taking into account the fact that $\mathbf{z}^\circ(0) = \mathbf{z}^\circ(T_0)$, $y_i^1(0) = y_i^1(T_0)$ $(i = 1, \ldots, 2n - 1)$, $z_{2n}^\circ(0) \neq 0$, and using the above relation we find that $y_{2n}^1(T_0) = y_{2n}^1(0) = 0$. Therefore $\mathbf{y}^1(T_0) = \mathbf{y}^1(0)$ which means that $\mathbf{y}^1(t)$ is a periodic solution of Eq.(1.2). Since $y_{2n}^1(0) = 0$, $y_{2n}^\circ(0) \neq 0$, the solutions $\mathbf{y}^1(t)$ and $\mathbf{y}^\circ(t)$ are linearly independent, which contradicts the assumption on the uniqueness of the periodic solution of (1.2). Consequently, $\det B_1 \neq 0$ and the $\operatorname{rank} B = 2n - 1$: this proves the first assertion of the theorem.

Let us now assume that the multiplicity of the unit multiplicator is $k = 2$, and the simple elementary divisors correspond to this multiplicator (otherwise we should have the same case as above). Suppose $\operatorname{rank} B < 2n-1$, then $\det B_1 = 0$; we shall show that $\operatorname{rank} B_1 = 2n-2$. In fact, for $\operatorname{rank} B_1 < 2n-2$, the equation $B_1 \mathbf{y} = 0$ has no less than two linearly independent solutions. As can be seen from the argument above, to these solutions there correspond the periodic solutions of (1.2), linearly independent with $\mathbf{y}^\circ(t)$, provided the total number of the periodic solutions is two. Therefore, the matrix B_2, obtained from B_1 by crossing out a certain row and column (to be specific, the column should be the last), is non-singular.

Since $\operatorname{rank} B < 2n-1$, the determinant of matrix B_3 obtained by crossing out the last column is zero. Consequently, the equation $B_3\mathbf{y} = 0$ has the non-trivial solution $\mathbf{a} = (c_0, \mathbf{c})'$, where $\mathbf{c} = (c_1, \ldots, c_{2n-2})$. Here $c_0 \neq 0$, otherwise the solution of the equation $B_2\mathbf{y} = 0$ would have a non-trivial solution $\mathbf{y} = c'$, which is impossible because $\det B_2 \neq 0$.

The identity $B_3\mathbf{a} = 0$ can be expressed as

$$c_0 \mathbf{y}_T{}^\circ + Y_{2n-1,2n-2}\mathbf{c}' = (\mathbf{c}, 0)', \quad \mathbf{y}_T{}^\circ = (y_1(T_0), \ldots, \mathring{y}_{2n-1}(T_0))' \tag{1.9}$$

where $Y_{2n-1,2n-2}$ is the matrix obtained from $Y(T_0)$ by crossing out the last row and the last two columns.

Let us assume that

$$\mathbf{y}^2 = c_0\mathbf{y}^\circ(T_0) + Y(T_0)\mathbf{y}^1, \quad \mathbf{y}^1 = (\mathbf{c}, 0, 0) \tag{1.10}$$

As follows from (1.9), $y_i{}^2 = y_i{}^1$ $(i = 1, \ldots, 2n-1)$. By virtue of (1.8), $(\mathbf{z}^\circ(0), \mathbf{y}^1) = (\mathbf{z}^\circ(T_0), Y(T_0)\mathbf{y}^1)$, and therefore from (1.10) and (1.5) we obtain

$$(\mathbf{z}^\circ(0), \mathbf{y}^1) = (\mathbf{z}^\circ(T_0), \mathbf{y}^2 - c_0\mathbf{y}^\circ(T_0)) = (\mathbf{z}^\circ(T_0), \mathbf{y}^2)$$

Since $y_i{}^2 = y_i{}^1$ $(i = 1, \ldots, 2n-1)$, $\mathbf{z}^\circ(T_0) = \mathbf{z}^\circ(0)$, $z_{2n}{}^\circ \neq 0$, we have $y_{2n}{}^2 = y_{2n}{}^1 = 0$. Thus $\mathbf{y}^2 = \mathbf{y}^1$, that is

$$\mathbf{y}^1 = Y(T_0)\mathbf{y}^1 + c_0\mathbf{y}^\circ(T_0), \quad c_0 \neq 0 \tag{1.11}$$

In view of the equation $\mathbf{y}^\circ(T_0) = Y(T_0)\mathbf{y}^\circ(T_0)$, this relation shows that the vectors $\mathbf{y}^\circ(T_0)$ and $\mathbf{y}^1$ belong to a cyclic subspace of the matrix $Y(T_0)$ which corresponds to the eigenvalue $\rho = 1$. However, this is impossible since, by the above assumption, the corresponding elementary divisors are simple.

Therefore, the assumption that $\operatorname{rank} B < 2n-1$ leads to a contradiction. Thus, $\operatorname{rank} B = 2n-1$, and this ensures the existence and uniqueness of a single-parameter family of the solutions $\alpha_i(s)$, and $T(s)$ of system (1.6), and therefore of the corresponding family $x_i(t, s)$ $(x_i(0, s) = \alpha_i(s))$ of periodic solutions of system (1.1). The theorem is proved.

Setting $\alpha_i = \alpha_i(s)$ and $T = T(s)$ in relations (1.6), and differentiating them with respect to s, we obtain

$$\boldsymbol{\alpha}_s = Y(T)\boldsymbol{\alpha}_s + \mathbf{x}^\cdot(T)T_s \tag{1.12}$$

$$\boldsymbol{\alpha}_s = \left(\frac{d\alpha_1(s)}{ds}, \ldots, \frac{d\alpha_{2n}(s)}{ds}\right), \quad T_s = \frac{dT(s)}{ds}$$

If the T_0-periodic solution $\mathbf{y}^\circ(t)$ is unique then $T_s(0) \neq 0$. In fact, for $T_s(0) = 0$, by virtue of (1.12) and because of the conditions $y_{2n}{}^\circ(0) \neq 0$, $(\alpha_{2n})_s = C_s = 0$, the solution $\mathbf{y}(t) = Y(t)\boldsymbol{\alpha}_s$ is a T_0-periodic and linearly independent solution $\mathbf{y}^\circ(t)$. If for $k = 2$ Eq. (1.2) has two periodic solutions, then $T_s(0) = 0$. In fact, for $T_s(0) \neq 0$ the vectors $\boldsymbol{\alpha}_s$ and $\mathbf{y}_0(T_0)$ by virtue of (1.12) form a cyclic subspace of the matrix $Y(T_0)$ which corresponds to the multiplicator $\rho = 1$, and this is impossible in view of the simplicity of the elementary divisors.

Let us show that for $k = 2$, the quantity H can be taken as the parameter s. The vectors $\mathbf{y}^\circ(0)$ and $\boldsymbol{\alpha}_s$ form a root subspace $\mathbf{a}_1, \mathbf{a}_2$ of the matrix $Y(T_0)$, corresponding to the eigenvalue $\rho = 1$. Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be the corresponding root subspace of the matrix $Y(T_0)'$. As we know (see /5/), $\Delta = \det\|(\mathbf{a}_p, \mathbf{b}_q)\|_{p,q=1}^2 \neq 0$. Since the monodromy matrix of Eq. (1.4), $Z(T_0) = (Y(T_0)')^{-1}$ and $Z(T_0)$ and $Z(T_0)^{-1}$ have the same eigenvectors, we have $Y(T_0)^1\mathbf{z}^\circ(0) = Z(T_0)^{-1}\mathbf{z}^\circ(0) = \mathbf{z}^\circ(0)$, therefore we can take $\mathbf{b}_1 = \mathbf{z}^\circ(0)$. Taking into account (1.5), we find $\Delta = -(\boldsymbol{\alpha}_s, \mathbf{z}^\circ(0))(\mathbf{y}^\circ(0), \mathbf{b}_2) \neq 0$, hence $(\boldsymbol{\alpha}_s, \mathbf{z}^\circ(0)) = dH(\mathbf{x}(0, s))/ds \neq 0$. Therefore the quantity H can serve as a parameter which determines the family of solutions in question.

Note. In proving the theorem we have ignored the Hamiltonian form of system (1.1). Therefore the system is valid for autonomous systems of general form, which have integral (1.3) (in particular, for Lyapunov systems). We also note that any system $\mathbf{x}^\cdot = f(\mathbf{x}, \beta)$ containing the parameter β can be reduced to the form indicated by including β in some of the variables.

2. Before we discuss the basic findings, we shall comment on the stability of periodic solutions. In those non-linear systems which have a unique first integral, the non-simple elementary divisors correspond, as a rule, to the multiplicator $\rho = 1$. For this reason, Eq. (1.2) has a solution of the form $\mathbf{y}^1(t) + t\mathbf{y}^\circ(t)$ and, therefore, the solution $\mathbf{x}^\circ(t)$ is Lyapunov-unstable. The necessary condition of the orbital stability of $\mathbf{x}^\circ(t)$ is the boundedness of the

remaining solutions. This certainly occurs if all the multiplicators of (1.2) lie on the unit circle and, with the exception of a double multiplicator $\rho = 1$, are definite, i.e. there are no coinciding multiplicators of different kind among them. We shall describe the solution $\mathbf{x}^\circ(t)$ for which these conditions are satisfied as orbitally stable to a first approximation.

We assume that $\mathbf{x} = 0$ is the equilibrium position of system (1.1), and $(H_\mathbf{x}(0) = 0)$, $A(0) = H_{\mathbf{xx}}(0)$ is a fixed-sign matrix (without loss of generality, positive definite). By this condition the eigenvalues of the matrix $J^{-1}A(0)$ are imaginary (see /5/); we denote them by $\pm i\omega_k^\circ$ $(k = 1, \ldots, n;\ 0 < \omega_i \leqslant \omega_{i+1})$. If $\omega_i/\omega_j \neq m$ for a certain j $(i = 1, \ldots, n;\ j \neq i;\ m$ is an integer), then in accordance with the Lyapunov theorem (see /1/) in a sufficiently small neighbourhood of the origin of coordinates there exists a unique single-parameter family of periodic solutions $\mathbf{x}^j(t, s)$ with periodic $T_j(s)$, such that $\mathbf{x}^j(t, s) \to 0$, $T_j(s) \to 2\pi/\omega_j^\circ$ when $s \to 0$.

Let Ω be a specified bounded domain, and $\mathrm{x} = 0$ its inner point, system (1.1) not having other equilibrium positions in Ω. Below we find the sufficient conditions for the family $\mathbf{x}^j(t, s)$ to be continued, in a unique way, with respect to s, to the boundary $\partial\Omega$ of the domain Ω, i.e. $\mathbf{x}^j(t, s) \in \Omega$ when $s \in (0, s_*)$, $\mathbf{x}(t_*, s_*) \in \partial\Omega$ for certain t_* and s_*. We note that the algorithms for a numerical search for periodic solutions of a Hamiltonian system by the method of continuation with respect to a parameter were developed in /6, 7/ and other publications.

Let A_- and A_+ by symmetrical positive definite constant matrices which satisfy the inequality

$$A_- < A(\mathbf{x}) < A_+ \quad \text{for} \quad \mathbf{x} \in \Omega \tag{2.1}$$

As usual, the latter means that $(A_-\mathbf{c}, \mathbf{c}) < (A(\mathbf{x})\mathbf{c}, \mathbf{c}) < (A_+\mathbf{c}, \mathbf{c})$ for any vector $\mathbf{c} \neq 0$. We denote the eigenvalues of the matrices $J^{-1}A_-$ and $J^{-1}A_+$ by $\pm i\omega_k^-$ and $\pm i\omega_k^+$

Theorem 2. If for a certain j,

$$m \overline{\in} \left[\frac{\omega_i^- + \omega_k^-}{\omega_j^+},\ \frac{\omega_i^+ + \omega_k^+}{\omega_j^-}\right] \tag{2.2}$$

$$i, k = 1, \ldots, n;\ m = 1, 2, \ldots;\quad k \neq j \quad \text{for} \quad i = j$$

then the family $\mathbf{x}^j(t, s)$ is, in a unique way, continuable in s to the boundary of the domain Ω. The corresponding period $T_j(s)$ satisfies the inequality

$$T_j^+ < T_j(s) < T_j^-;\quad T_j^+ = \frac{2\pi}{\omega_j^+},\quad T_j^- = \frac{2\pi}{\omega_j^-} \tag{2.3}$$

For any $s \in (0, s_*]$ the solution $\mathbf{x}^j(t, s)$ is orbitally stable to a first approximation.

Proof. By (2.1) and (2.2) we have $\omega_i^- < \omega_i^\circ < \omega_i^+$, $\omega_j^\circ \neq \omega_i^\circ/m$, therefore for small s the family $\mathbf{x}^j(t, s)$ indicated in the theorem exists. First we shall show that if $\mathbf{x}^j(t, s)$ is continuable in s to a certain $(0, s_1]$ then the corresponding period $T_j(s)$ satisfies the inequality (2.3).

Consider the selfconjugate boundary value problem

$$J\mathbf{y}^\cdot = [A_- + \lambda(R(t) - A_-)]\,\mathbf{y},\quad \mathbf{y}(0) = \mathbf{y}(T) \tag{2.4}$$

where $R(t)$ is a symmetrical positive definite matrix.

We denote by $\pm i\omega_k(\lambda)$ the eigenvalues of the matrix $J^{-1}[A_- + \lambda(A_+ - A_-)]$. When λ grows from zero to unity, $\omega_k(\lambda)$ increases monotonically from ω_k^- to ω_k^+. For $R = A_+$ the positive eigenvalues of problem (2.4) are the roots of the equations $2\pi m/T = \omega_k(\lambda)$ $((k = 1, \ldots, n;\ m = 1, 2, \ldots)$. By (2.2), $\omega_j(\lambda) \neq m^{-1}[\omega_k^-, \omega_k^+]$ for $\lambda \in (0, 1)$, $k \neq j$; $\omega_j^- > \omega_j^+/2$. Therefore, when $R = A_+$, we have either $T = T_j^+$ or $T = T_j^-$, and the boundary value problem (2.4) has no eigenvalues on (0.1). Since $A(\mathbf{x}_j(t, s)) < A_+$, and for an increase in $R(t)$ the positive eigenvalues decrease (see /8/), then for $R = A(\mathbf{x}^j(t, s))$, $T = T_j^+$ or $T = T_j^-$ the eigenvalues $\lambda_i \neq 1$. Since $\mathbf{y} = \mathbf{x}^\cdot(t, s)$ satisfies Eq.(1.2), for $R(t) = A(\mathbf{x}^j(t, s))$ and $T = T_j(s)$ problem (2.4) has an eigenvalue $\lambda = 1$. Thus, $T_j(s) \neq T_j^+$, $T_j(s) \neq T_j^-$, i.e. for $s \in (0, s_1]$ inequality (2.3) is not violated.

For $R = A_+$ the multiplicators of the first and second kind of Eq.(2.4) are $r_k^1(\lambda) = \exp(i\omega_k(\lambda)T)$ and $r_k^2(\lambda) = \exp(-i\omega_k(\lambda)T)$ respectively; for $\lambda \in [0,1]$ they are on the arcs $\Gamma_k^1 = (r_k^1(0), r_k^1(1))$ and $\Gamma_k^2 = (r_k^2(0), r_k^2(1))$ of the unit circle. By virtue of (2.2) and (2.3) only the arcs Γ_j^1 and Γ_j^2 have common points, and therefore for $\lambda \in [0, 1]$ the multiplicators of a different kind $r_p^1(\lambda)$ and $r_q^2(\lambda)$ are not identical, with the exception of $r_j^1(\lambda)$ and $r^2{}_j(\lambda)$. The same argument holds for the multiplicators $\rho_p^1(\lambda)$, $\rho_q^2(\lambda)$ of (2.4) when $R = A(\mathbf{x}^j(t, s))$.

In fact, when λ increases the multiplicators $\rho_p^1(\lambda)$ and $\rho_q^2(\lambda)$ $(\rho_p^1(0) = r_p^1(0),\ \rho_q^2(0) = r_q^2(0))$ move along arcs Γ_p^1 and Γ_q^2, anticlockwise and clockwise respectively (see /8/). Let us assume that for $\lambda \leqslant 1$ they meet; then for a certain $\lambda_* < 1$ the multiplicator ρ_p^1 or ρ_q^2 is at the point $\rho = \rho_*$, not on the arcs Γ_i^1, Γ_i^2 $(i = 1, \ldots, n)$. Consequently, if $R = A(\mathbf{x}^j(t, s))$, the selfconjugate boundary value problem for Eq.(2.4) with the boundary conditions $\mathbf{y}(T_j) = \rho_*\mathbf{y}(0)$ has the eigenvalues $\lambda_* \in (0, 1)$. Because, as R increases the positive eigenvalues decrease, and at the same time $\lambda_i \neq 0$ by virtue of (2.3), for $R = A_+$ this problem also has eigenvalues

$\lambda_k \in (0, 1)$, i.e. one of the multiplicators $r_i^1(\lambda_k)$, $r_i^2(\lambda_k)$ equals ρ_*. However, this is impossible because $r_i^1(\lambda) \in \Gamma_i^1$ $r_i^2(\lambda) \in \Gamma_i^2$ for $\lambda \in [0, 1]$.

Thus, the multiplicators ρ_p^1 and ρ_q^2 $(p, q = 1,\ldots, n;\ p, q \neq j)$ of Eq.(1.2) which corresponds to the solution $\mathbf{x}^j(t, s)$ lie on the unit circle and are definite. Consequently, the multiplicity of the multiplicator $\rho = 1$ equals two, and by Theorem 1 the solution $\mathbf{x}^j(t, s)$ is locally continuable in s. For this reason, when $\mathbf{x}^j(t, s) \in \Omega$ only that value of $s = s_*$ can be a limit value for which $\mathbf{x}^j(t, s) \to \mathbf{c}$ as $s \to s_*$. By this condition, system (1.1) in the domain Ω has a unique equilibrium position $\mathbf{x} = 0$, and therefore $\mathbf{c} = 0$. Since only the numbers $T_i^\circ = 2\pi/\omega_i^\circ$ $(i = 1,\ldots, n)$, can serve as bifurcation points of the equilibrium position on the T axis, by (2.2) and (2.3), as $s \to s_*$ we have $T_j(s) \to T_j^\circ$. But then together with $x_j(t, s)$ there exists a single-parameter family $\mathbf{x}^*(t, s) = \mathbf{x}^j(t, s_* - s)$ such that $\mathbf{x}^*(t, s) \to 0$, $T(s) \to 2\pi/\omega_j^\circ$ as $s \to 0$. This contradicts the assertion of Lyapunov's theorem on the uniqueness of such a family, and it proves that $\mathbf{x}^j(t, s)$ is continuable in s up to the boundary of the domain Ω.

The above argument shows that the multiplicators of Eq.(1.2) are definite when $\mathbf{x} = \mathbf{x}^j(t, s)$, with the exception of $\rho_j^1 = \rho_j^2 = 1$. For this reason $\mathbf{x}^j(t, s)$ is orbitally stable to a first approximation. The theorem is completely proved.

3. Let us discuss the theorem in more detail. Suppose that for $\mathbf{x} \in \Omega$ we have $H(0) = 0$, $H(\mathbf{x}) \leqslant M$. Then, provided that (2.2), any periodic solution $\mathbf{x}(t) \in \Omega$ with period $T \in (T_j^+, T_j^-)$ belongs to the family $\mathbf{x}^j(t, s)$.

In fact (see the proof of Theorem 2), provided that (2.2) the multiplicity of the multiplicator $\rho = 1$ which corresponds to any solution $\mathbf{x}(t)$ with period $T \in (T_j^+, T_j^-)$ equals two. In conformity with Theorem 1, $\mathbf{x}(t)$ belongs to a single-parameter family of periodic solutions, and the quantity H can be taken as a parameter. Continuing $\mathbf{x}(t, H)$ in H up to $H = 0$ (here, as can be seen from the proof of Theorem 2, the inequality (2.3) is maintained, and $T(H) \to T_j^\circ$ as $H \to 0$), we can find by means of Lyapunov's theorem that for small H, the family $\mathbf{x}(t, H)$ is identical with the family $\mathbf{x}^j(t, H)$ indicated in the theorem. Because of the uniqueness of the continuation, these families are identical for all $H \leqslant M$.

The number of the indices $j \in [1,\ldots, n)$ for which condition (2.2) is satisfied, yields the lower estimate of the number of periodic solutions which lie on any isoenergetic surface $H(\mathbf{x}) = H \leqslant M$.

If (2.2) is satisfied for all $\mathbf{x} \in R^{2n}$, then $\mathbf{x}^j(t, X)$ can be continued in H on $(0, \infty)$. At the same time for any H the solution with period $T \in (T_j^+, T_j^-)$ is unique and belongs to the family $\mathbf{x}^j(t, H)$.

As shown in /9/, provided that

$$\omega \overline{\in} [\omega_k, \omega_k^+]/m,\ k = 1,\ldots, n;\ m = 1,2,\ldots \tag{3.1}$$

system (1.1) has a unique solution with period $T = 2\pi/\omega$. Since $\mathbf{x}(t) \equiv 0$ is such a solution, oscillations with period $2\pi/\omega$ are not possible (as a corollary, oscillations with period $T \leqslant 2\pi/\omega_n^+$ are also impossible). Hence it follows in particular that under the condition (2.2), $T_j(s)$ is the minimum period of the solutions $\mathbf{x}^j(t, s)$ because, for $\omega \in p(\omega_j^-, \omega_j^+)$ condition (3.1) holds ($p > 1$ is an integer).

On the other hand, if

$$m \overline{\in} p\left[\frac{\omega_i^- + \omega_k^-}{\omega_j^+},\ \frac{\omega_i^+ + \omega_k^+}{\omega_j^-}\right],\quad i, k = 1,\ldots, n; \tag{3.2}$$
$$i \neq j \ \text{ for } k = j;\quad m = 1, 2,\ldots$$

then any periodic solution $\mathbf{x}(t)$ with period $T \in p(T_j^+, T_j^-)$ has a minimum period $T_{\min} = T/p$, and therefore it belongs to the family $\mathbf{x}^j(t, s)$.

In fact, (see the proof of Theorem 2), under the condition (3.2) the multiplicity of the multiplicator $\rho = 1$ of (1.2) is 2; therefore any solution $\mathbf{x}(t)$ is continuable in H up to $H = 0$, and at the same time the corresponding period $T(H) \in p(T_j^+, T_j^-)$. But, by virtue of (3.2), $\omega_i^\circ \overline{\in} [\omega_j^-, \omega_j^+]$ for $i \neq j$, and consequently $T(H) \to 2\pi p/\omega_j^\circ$, and $T_{\min}(H) \to 2\pi/\omega_j^\circ$ as $H \to 0$, that is $\mathbf{x}(t)$ belongs to the family $\mathbf{x}^j(t, s)$.

We assume that $H(\mathbf{x})$ is an even function with respect to the generalized momenta, i.e.

$$H(x_1,\ldots, x_n, x_{n+1},\ldots, x_{2n}) = H(x_1,\ldots, x_n, -x_{n+1},\ldots, -x_{2n}) \tag{3.3}$$

Then, together with the solution $\mathbf{x}(t) = (x_1(t),\ldots, x_{2n}(t))'$, the function $\mathbf{x}^*(t) = (x_1(-t),\ldots, x_n(-t), -x_{n+1}(-t), \ldots, -x_{2n}(-t))'$ also satisfies system (1.1). If the periodic solution $\mathbf{x}(t)$ is unique (to within a shift in t), then for a certain h the identity $\mathbf{x}^*(t) = \mathbf{x}(t + h)$ should hold; hence $x_i(\tau) = x_i(-\tau)$, $x_{i+n}(\tau) = -x_{i+n}(-\tau)$, where $\tau = t - h/2$. Consequently, provided that (2.2) and (2.3), and with an appropriate choice of the reference point, we have

$$x_i^j(t, s) = x_i^j(-t, s),\quad x_{i+n}^j(t, s) = -x_{i+n}^j(-t, s),\quad i = 1,\ldots, n \tag{3.4}$$

Let us assume that $H(\mathbf{x})$ is an even function of the coordinates and momenta

$$H(\mathbf{x}) = H(-\mathbf{x}) \tag{3.5}$$

In this case the theorem remains valid, if we replace (2.2) by the weaker condition,

$$2m \,\overline{\in}\, \left[\frac{\omega_i^- + \omega_k^-}{\omega_j^+},\ \frac{\omega_i^+ + \omega_k^+}{\omega_j^-}\right], \quad i, k = 1, \ldots, n; \quad i \neq j \tag{3.6}$$
$$\text{for } k = j; \quad m = 1, 2, \ldots$$

By this condition, $\omega_j^\circ \neq \omega_i^\circ/m$, and therefore in a sufficiently small neighbourhood of the origin, $\mathbf{x}^j(t, s)$ exists. Provided that (3.5), the function $-\mathbf{x}^j(t, s)$ also satisfies system (1.1); hence, taking into account the uniqueness of $\mathbf{x}^j(t, s)$ we find that for a certain h, $\mathbf{x}^j(t+h, s) = -\mathbf{x}^j(t, s)$. Consequently, $\mathbf{x}^j(t+2h, s) = \mathbf{x}^j(t, s)$, $h = T/2$; that is

$$\mathbf{x}^j(t + T/2, s) = -\mathbf{x}^j(t, s) \tag{3.7}$$

It follows from (3.5) and (3.7) that $A(\mathbf{x}) = A(-\mathbf{x})$, $A_j(t) = A(\mathbf{x}^j(t, s)) = A(\mathbf{x}^j(t + T/2, s))$, i.e. the smallest period of the matrix $A_j(t)$ is $T/2$, and therefore the multiplicators of (1.2), ρ_i, equal the eigenvalues of the matrix $Y(T/2)$. We find, in a way similar to the proof of Theorem 2, that under condition (3.6) all multiplicators lie on the unit circle and, with the exception of $\rho = -1$, they are all definite. Since the eigenvalues of the matrix $Y(T)$ are $\rho_i^2 (Y(T) = Y(T/2)^2)$, the multiplicity of its unique eigenvalue is two, which in conformity with Theorem 1 ensures the uniqueness of continuing $\mathbf{x}^j(t, s)$ in s in the domain Ω. In turn, the latter guarantees relation (3.7).

4. As an example, let us look into the oscillations of a string with limped masses. Assuming that there is no longitudinal shift of mass, we shall find the Hamiltonian function

$$H = \frac{EF}{2} \sum_{i=0}^{n} \frac{1}{l_i} \left(\frac{T_0 l_i}{EF} + \sqrt{z_i^2 + l_i^2} - l_i\right)^2 + \frac{1}{2} \sum_{i=1}^{n} \frac{x_{i+n}^2}{m_i}$$
$$z_i = x_{i+1} - x_i \quad (i = 0, 1, \ldots, n-1), \quad z_n = x_n, \quad x_0 = 0$$

where $x_1, \ldots, x_n$ are the transverse shifts of masses $m_1, \ldots, m_n$; $x_{n+1}, \ldots, x_{2n}$ are the corresponding momenta; $l_0, \ldots, l_n$ are the lengths of the successive segments; E is the elastic modulus, F is the cross-sectional area, and T_0 denotes the initial tension of the string.

We reduce the Hessian of the Hamilton function to the form

$$(A(\mathbf{x})\mathbf{c}, \mathbf{c}) = \sum_{i=0}^{n} \frac{EF}{l_i} \left(\frac{(k-1) l_i^3}{\sqrt{(z_i^2 + l_i^2)^3}} + 1\right) b_i^2 + \sum_{i=1}^{n} \frac{c_{n+i}^2}{m_i} \tag{4.1}$$
$$b_i = c_{i+1} - c_i \quad (i = 0, 1, \ldots, n-1), \quad b_n = c_n, \quad c_0 = 0$$

Clearly, when $k > 1$, to obtain the form $(A_+ \mathbf{c}, \mathbf{c})$, or $(A_- \mathbf{c}, \mathbf{c})$ it is sufficient to set in (4.1) $z_i^2 = 0$ or $z_i^2 = z_{i0}^2 = \max z_i^2$ when $x_{i+1}, x_i \in \Omega$ $(i = 0, 1, \ldots, n)$; and for $k < 1$ just the opposite. Let ω_i° and ω_i be the corresponding frequencies, then $\omega_i^+ = \omega_i^\circ$, $\omega_i^- = \omega_i$ for $k > 1$, $\omega_i^- = \omega_i^\circ$, $\omega_i^+ = \omega_i$ for $k < 1$. The quantities ω_i° equal the frequencies of small natural oscillations, therefore for $k > 1$ the periods $T_j(H)$ of the solutions $x^j(t, H)$ are longer, and for $k < 1$ shorter, than the corresponding periods of small oscillations, T_j°; for $k = 1$ the system becomes linear. We note that physically k is the relative elongation of the string due to the tension T_0.

Since $H(\mathbf{x}) = H(-\mathbf{x})$, due to condition (3.6), in the domain Ω there exists the family of periodic solutions $\mathbf{x}^j(t, H)$ as $\mathbf{x}^j(t, H) \to 0$ and $T_j(H) \to 2\pi/\omega_i^\circ$ as $H \to 0$. This family is orbitally stable to a first approximation, and satisfies relations (3.4) and (3.7).

When the domain Ω increases, the corresponding frequencies $\omega_i(\Omega)$ then to ω_i° and for this reason the number of indices j for which the condition (3.6) holds, increases.

If $\Omega = R^{2n}$, assuming in (3.1) that $z_i \to \infty$, we find that $\omega_i = \omega_i^\circ/\sqrt{k}$. Let us denote by k_j^- and k_j^+ the limit values of k for which the condition (3.6) is violated (thus, for $k_j^- < k < k_j^+$ the solution $\mathbf{x}^j(t, H)$ is continuable to any H). Clearly, k_j^+ equals the following number nearest to unity from the right; and k_j^-, that from the left:

$$\left(\frac{\omega_i^\circ + \omega_k^\circ}{2\omega_j^\circ m}\right) \text{ or } \left(\frac{2\omega_j^\circ m}{\omega_i^\circ + \omega_k^\circ}\right)$$
$$i, k = 1, \ldots, n; \quad i \neq j \quad \text{for } k = j; \quad m = 1, 2, \ldots$$

Therefore, $k_j^- = 1/k_j^+$. The relation $k_n^- = (\omega_{n-1}^\circ + \omega_n^\circ)^2 (2\omega_n^\circ)^{-2}$. is computed directly.

For example, let $n = 4$, $l_i = l_0$, $m_i = m$ $(i = 1, \ldots, 4)$, $2T_0^{1/2}(l_0 m)^{-1/2} = 1$. Then $\omega_i^\circ = \sin k_i$, $k_i = \pi i/2(n + 1)$, hence $\omega_1^\circ = 0.3090$, $\omega_2^\circ = 0.5878$, $\omega_3^\circ = 0.8090$, $\omega_4^\circ = 0.9512$. Corresponding calculations yield $k_1^- = 0.963$, $k_1^+ = 1.039$, $k_2^- = 0.709$, $k_2^+ = 1.411$, $k_3^- = 0.846$, $k_3^+ = 1.183$, $k_4^- = 0.857$, $k_4^+ = 1.167$.

Notice that in the case of a finite domain Ω the solution $\mathbf{x}^j(t, H)$ is certainly continuable to the boundary of Ω if the frequencies $\omega_i(\Omega) > \omega_i^\circ/\sqrt{k_j^+}$ for $k > 1$ or $\omega_i(\Omega) < \omega_i^\circ/\sqrt{k_j^-}$ for $k < 1$ $(i = 1, \ldots, n)$.

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STABILITY OF THE UNIFORM ROTATION OF A GYROSTAT ROUND THE VERTICAL MAIN AXIS ON AN ABSOLUTELY SMOOTH HORIZONTAL PLANE*

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The motion of a gyrostat on an absolutely smooth plane is discussed. A Hamilton function which gives the canonical equations of motion is obtained. This admits of particular solutions, namely uniform rotations round a vertical axis which are identical with that of the uniform rotations of the rotor. A transition to a system with two degrees of freedom is realized, and the expansion of the Hamiltonian in the vicinity of the corresponding position of equilibrium, with an accuracy to within fourth-order terms, is obtained. In the region of admissible values of the parameters the domain of the necessary stability conditions, and the domains where the Hamiltonian functions are of fixed sign and alternating, are examined. In those cases where the Hamiltonian is not fixed sign, its normalization is performed, both a non-resonance situation and resonances of the first, second and fourth order being considered. The sufficient conditions for stability of uniform gyrostat rotation in terms of constraints on the coefficients of normal forms are obtained. For a clear interpretation of the results, special cases where the values of all the parameters except two are fixed, are given. The plane domain of the necessary stability conditions and resonance curves are constructed, and using computer results stability on the curves is discussed.

The stability of uniform rotations of a heavy solid around the vertical principal and minor axes on an absolutely smooth, and on an absolutely rough horizontal plane, and also on a plane with viscous friction is discussed in /1—4/. The stability of uniform rotations of a gyrostat round the vertical principal axis on absolutely smooth and absolutely rough horizontal planes was considered in /5, 6/. Investigations of the motion of a solid on an absolutely rough plane, the body being perturbed with respect to rotation round the principal axis (in particular with respect to the steady position of equilibrium), are described in

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